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# Existence of dynamics for a 1D NLS equation perturbed with a generalized point defect

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**Abstract**

In the present paper we study the well-posedness for the one-dimensional cubic NLS perturbed by a generic point interaction. Point interactions are described as the 4-parameter family of self-adjoint extensions of the symmetric 1D Laplacian defined on the regular functions vanishing at a point, and in the present context can be interpreted as localized defects interacting with the NLS field. A previously treated special case is given by an NLS equation with a $\delta$ defect which we generalize and extend, as far as well-posedness is concerned, to the whole family of point interactions. We prove existence and uniqueness of the local Cauchy problem in strong form (initial data and evolution in the operator domain of point interactions), weak form (initial data and evolution in the form domain of point interactions) and $L^2(\mathbb{R})$. Conservation laws of mass and energy are proved for finite energy weak solutions of the problem, which imply global existence of the dynamics. A technical difficulty arises due to the fact that a power nonlinearity does not preserve the form domain for a subclass of point interactions; to overcome it, a technique based on the extension of resolvents of the linear part of the generator to maps between a suitable Hilbert space and the energy space is devised and estimates are given which show the needed regularization properties of the nonlinear flow.

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## 1. Introduction

The present paper is devoted to the well-posedness of a nonlinear Schrödinger (NLS) equation with a point defect in dimension 1. The Schrödinger equation bears a cubic nonlinearity, and the defect is described by the general point interaction in dimension 1. To be precise, the equation to be studied is given by

$$\begin{cases} \mathrm{i}\partial_t \psi(t) = H\psi(t) + \lambda|\psi(t)|^2\psi(t) \\ \psi(0) = \psi_0 \end{cases} \tag{1.1}$$

or in weak form [21]

$$\psi(t) = \mathrm{e}^{-\mathrm{i}Ht}\psi_0 - \mathrm{i}\lambda \int_0^t \mathrm{d}s\, \mathrm{e}^{-\mathrm{i}H(t-s)}|\psi(s)|^2\psi(s) \tag{1.2}$$

where $\psi_0$ represents the initial data and $H$ is the Hamiltonian operator generating a point interaction at the origin. Point interactions are singular perturbations of the Laplace operator; restricting the Laplacian on the set of regular functions vanishing at a point gives a symmetric but not self-adjoint operator; its self-adjoint extensions are by definition point interactions [6, 7]. In one dimension (1D) they form a four-parameter family of s.a. operators [5], and they describe, in an effective way, a variety of situations relevant to the dynamics and scattering of quantum particles acted on by strongly concentrated potentials. In dimension 1, their domains are characterized by suitable boundary conditions, which apart from the standard ones (i.e. Dirichlet, Neumann, Robin, one sided or two sided) comprise analogous boundary conditions involving the jump of the functions ($[\psi]$) or their derivatives at the point where the interaction takes place. Among the main non-trivial examples are the well-known $\delta$ interaction (where the boundary condition is $[\psi'](0) = \alpha\psi(0)$) often called Fermi pseudopotential in the physics literature, or the $\delta'$ interaction (roughly speaking $[\psi](0) = \alpha\psi'(0)$, see [16] for details). In the case of NLS with $\delta$ interaction there exists a certain amount of literature, physical, numerical and mathematical, concerning the existence of stationary states [9–11, 29], the asymptotic behavior in time [22, 23], and the reduced dynamics on the stable soliton manifold [15, 20]. Little is known for the $\delta'$ interaction, and nothing in the generic case.

Quite generally, equation (1.1) is a prototype of the interactions of nonlinear waves propagating in media in which inhomogeneities are present. One possible physical interpretation of the model described by equation (1.1) is given by the interaction of a 1D Bose condensate with an impurity. To the right and left point of the perturbation the Bose condensate satisfies, as an effective equation in the limit of infinite bosons (see for the 1D case [1], for the 3D setting [17, 18, 25]), and the NLS equation (usually called the Gross–Pitaevskii equation) in this context. At the defect or impurity location a boundary condition establishes the nature of interaction, and gives the link between the two sides of the condensate. Bose condensates are quantum many-body systems which display a typical macroscopic behavior, and is measured in an effective way through the scattering length of the underlying two-body interaction. So the use of a point interaction in (1.1) is legitimate if the scale length at which the interaction takes place is far smaller than the characteristic scattering length of the Bose condensate. A second interpretation is of classical origin. It is well known that the propagation of an optical wave pulse in a nonlinear dispersive medium (such as an optical fiber) gives rise to a NLS equation for the evolution of the pulse envelope [8]. The presence of defects or junctions in the fiber can be modeled through boundary conditions, and in the simple and generally adopted case of 1D propagation along the fiber this corresponds to consider a point perturbation.

A different occurrence of an effective point ($\delta$) interaction is in the study of bimodal optical fibers; these devices are described by two coupled NLS which admit two-soliton solutions; in a typical situation the solitons are one narrow and one wider. At a formal level it turns out that in a suitable limit the pulse propagation is described by a single NLS and the effect of the narrower soliton can be represented by a $\delta$ interaction, at least as far as its influence on the dynamics of the wider one is concerned (see [11] and references therein). The previous applications acquire an additional interest due to the evidence, both on the numerical and rigorous side, of a certain persistence of the soliton behavior even in the presence of a breaking of translational invariance due to a $\delta$ interaction [22, 23]; in particular a fast soliton breaks into two pieces, one reflected and one transmitted, the relative amplitude of which being controlled by the scattering matrix of the $\delta$ interaction at least within times long with respect to the interaction time. This is a meaningful phenomenon quite different from orbital stability and indicates a peculiar robustness of the soliton solutions of NLS, even in the case of strong interaction with external perturbation.

Among the possible future perspectives, let us cite the analysis of the dynamics of the system investigated in [13], where the ground states of the stationary Maxwell–Schrödinger system in a bounded domain with a point defect are studied. In such a case the nonlinearity arises from a Hartree-type interaction, and is milder than in the usual NLS; nevertheless, the results and the techniques are interesting because they apply in dimension 3. To this aim, one could exploit the dispersive estimates for a linear 3D system with point interactions, given in [14].

As a final quotation, we mention the paper [26], which deals with well-posedness, direct and inverse scattering for a family of NLS with potential terms.

We are not concerned here with a rigorous justification (which is lacking) of the point interaction as an effective model of scatterer or junction in the NLS propagation phenomenon. We assume it as a plausible one and we proceed to show its existence and uniqueness in the small and in the large and qualitative properties, such as energy conservation for the whole family of point interactions. At a rigorous level, well-posedness of the problem (1.2) is well known for the case of $\delta$ interaction only (see [20, 23]). Here we stress that, the generic point interaction, which is considered as a potential, is *not* in $H^{-1}$. Actually, the energy domain of the problem is larger than $H^1$.

In the following, we give a brief summary of the results obtained in the paper and the techniques involved in the proofs.

In section 2, the definition and main properties of general point interactions are given. In particular, the form domain and quadratic form of the family are described completely (as far as we know this material is not published elsewhere).

In section 3, we state and prove local existence, uniqueness and blow-up alternative for strong solution to (1.1) (theorem 3.1). By strong solutions we mean solutions $t \to \psi(t)$ with values in the domain of $H$. Then we prove energy conservation for the strong flow as a consequence of which blow-up does not occur and maximal solutions are in fact global in time. The proof of theorem 3.1 is through a contraction in a suitable neighborhood of the initial point in the space $\mathcal{X} = C([0,T], D(H)) \cap C^1([0,T], L^2(\mathbb{R}^2))$. The main technical tools are: i) a bound on the $L^\infty$ norm on $\psi$ and its graph norm $\|H\psi\|_{L^2(\mathbb{R})} + \|\psi\|_{L^2(\mathbb{R})}$; ii) an integration by parts in the integral form of problem (1.1) to get a regularization of most singular terms.

In section 4 we treat local and global well posedness in the finite energy space, i.e. form domains, for the family of point interactions whose form domain is given by $H^1(\mathbb{R}^+) \oplus H^1(\mathbb{R}^-)$. Quite general results of local existence in which the linear part of the generator is self-adjoint are known in the literature (see theorem 3.9.9 in [12] and section 3.7). In such a case one obtains a local existence of a solution $\psi \in C([0,T], X) \cap C^1([0,T], X^\star)$, where $X$ is the form domain of the linear part of the generator, if uniqueness is known and a set of hypotheses are satisfied. Here we prefer to proceed in a direct way, because the verification of the hypotheses of the quoted general results for a part of the family of point interactions is not simpler than a direct proof. In section 5, we explicitly treat the delicate case in which a boundary condition is present in the definition of the energy space: this is indeed the case for a subclass of the family of point interactions, whose form domain is given by $Q_{\omega a} = \{\psi \in H^1(\mathbb{R}^+) \oplus H^1(\mathbb{R}^-) \,|\, \psi(0_+) = \omega a \psi(0_-), |\omega| = 1, a \in \mathbb{R}\backslash\{0, \pm 1\}\}$. In the first place the nonlinearity does not preserve the boundary condition. As a consequence, for these interactions there appear terms in the Duhamel formula that are difficult to immediately recognize as elements of the energy space. To get the relevant estimates, we prove that the resolvent of the linear part can be continuously extended to a suitable Banach space, larger than the dual of the energy spaces, with values still in the energy space. In the end, one obtains for NLS with arbitrary point interactions local and global well-posedness on the form domain of point interaction itself, stated and proved in theorem 5.10. The relevant properties of the duals of the energy spaces needed in the proofs, are given at the beginning of this section.

The last result (section 6) is the global well-posedness in $L^2(\mathbb{R})$. The proof is based on the dispersive behavior of the propagator associated with $H$ and Strichartz-type estimates on the same propagator. We obtain that the usual Duhamel map is a contraction in a neighborhood of the initial data in a space of the form $L^\infty_t L^2_x \cap L^q_t L^r_x$ for some couples $(q, r)$. Incidentally, the estimates are insensitive to the presence of stationary states.

In section 7 we give a brief summary on possible extensions of the result, and try to outline how to obtain them.

Our results extend to dynamics on graphs, which is the subject of a paper in preparation.

## 2. Preliminaries

### *2.1. Notation*

Here we fix some basic notation that we will use throughout the paper.

(1) The symbol $(\psi, \phi)$ denotes the scalar product in $L^2(\mathbb{R})$ between the functions $\psi$ and $\phi$, according to the definition

$$(\psi, \phi) := \int_{\mathbb{R}} \overline{\psi(x)}\phi(x)\, \mathrm{d}x.$$

(2) The symbol $\langle f, \psi\rangle_{X^\star,X}$ denotes the duality product between the functional $f \in X^\star$ and the vector $\psi \in X$.

(3) We denote by $\widehat{\psi}$ or $\mathcal{F}\psi$ the Fourier transform of the function $\psi \in L^2(\mathbb{R})$; the convention on the normalization is the one given (when meaningful) by

$$\widehat{\psi}(k) = (2\pi)^{-1/2} \int_{\mathbb{R}} \psi(x)\, \mathrm{e}^{-\mathrm{i}kx}\, \mathrm{d}x.$$

The symbol $\mathcal{F}$ will be used also to denote the Fourier transform in the distribution space.

(4) The symbols $\chi_\pm$ denote the characteristic function of the sets $\mathbb{R}^\pm$.

(5) Some particular functions will play an important role and we will need to use them frequently. Therefore, we define a notation for them:

$$\begin{aligned} \varphi_\pm(x) &= \chi_\pm(x)\, \mathrm{e}^{\mp x} \\ \varphi_\nu(x) &= \nu\varphi_+(x) + \varphi_-(x). \end{aligned} \tag{2.1}$$

Note that, as a particular case, $\varphi_1 = \mathrm{e}^{-|\cdot|}$. We will use also

$$\begin{aligned} \varphi^z_\pm(x) &= \chi_\pm(x)\, \mathrm{e}^{\mp\sqrt{z}x} \\ \varphi^z(x) &= \varphi^z_+(x) + \varphi^z_-(x) = \mathrm{e}^{-\sqrt{z}|x|} \end{aligned} \tag{2.2}$$

where $z \in \mathbb{C}$, $\mathrm{Re}\sqrt{z} > 0$.

(6) We often deal with functions of the type $\psi = \chi_+\psi_+ + \chi_-\psi_-$, with $\psi_\pm \in H^1(\mathbb{R})$. With a slight abuse of notation, we denote

$$\psi' = \chi_+\psi'_+ + \chi_-\psi'_-.$$

(7) The norm of $\psi$ in the space $L^p(\mathbb{R})$ is denoted by $\|\psi\|_p$, except for $p = 2$, in which case we omit the subscript. For any other space we explicitly refer to the space in the subscript.

*2.2. Point interactions in dimension 1: operators and forms*

By definition, the family of hamiltonian operators describing the dynamics of a particle in dimension 1 under the influence of a scattering center located at the origin is obtained as the set of self-adjoint extensions (s.a.e.) of the operator

$$\widehat{H}_0 = -\partial_x^2 \tag{2.3}$$

defined on the domain

$$D(\widehat{H}_0) = C_0^\infty(\mathbb{R}\backslash\{0\}). \tag{2.4}$$

Following [5] and [19], any s.a.e. of $\widehat{H}_0$ can be described in one of the two following ways.

- Given $\omega, a, b, c, d$ such that $|\omega| = 1$, $ad - bc = 1$, we define the s.a.e. $H_U$ as follows:

$$U = \omega\begin{pmatrix} a & b \\ c & d \end{pmatrix},$$
$$D_U := D(H_U) = \left\{\psi \in H^2(\mathbb{R}\backslash\{0\}), \begin{pmatrix} \psi(0+) \\ \psi'(0+) \end{pmatrix} = U\begin{pmatrix} \psi(0-) \\ \psi'(0-) \end{pmatrix}\right\}, \tag{2.5}$$
$$(H_U\psi)(x) = -\psi''(x), \qquad x \neq 0, \qquad \forall\, \psi \in D(H_U).$$

- Given $p, q \in \mathbb{R}\cup\{\infty\}$ we define the s.a.e $H_{p,q}$ as follows:

$$\begin{aligned} &D_{p,q} := D(H_{p,q}) = \{\psi \in H^2(\mathbb{R}\backslash\{0\}), \psi(0+) = p\psi'(0+), \psi(0-) = q\psi'(0-)\}, \\ &(H_{p,q}\psi)(x) = -\psi''(x), \qquad x \neq 0 \qquad \forall\, \psi \in D(H_{p,q}). \end{aligned} \tag{2.6}$$

Given the quantity $m := 1 - \inf\sigma(H) < \infty$, we introduce the norm $\|\psi\|_H := \|(H+m)\psi\|$ that endowes $D(H)$ with the structure of a Hilbert space.

In the following we investigate the problem of finding solutions to (1.1) in the form domain of the linear part of the generator. In order to do that we preliminarily recall such form domains (see e.g. [24]).

**Proposition 2.1.** *The quadratic forms associated with the self-adjoint extensions of $\widehat{H}_0$ are defined as follows.*

*(1) For the Hamiltonian $H_{0,0}$ the energy space is*

$$Q_0 := \{\psi \in H^1(\mathbb{R}), \psi(0) = 0\} \tag{2.7}$$

*and the form reads*

$$B_0(\psi) = \|\psi'\|^2. \tag{2.8}$$

*(2) For the Hamiltonian $H_{0,q}$, $q \neq 0$,*

$$Q_{0+} := \{\psi \in H^1(\mathbb{R}^+) \oplus H^1(\mathbb{R}^-), \psi(0+) = 0\} \tag{2.9}$$

*and*

$$B_{0,q}(\psi) = \|\psi'\|^2_{L^2(\mathbb{R}^+)} + \|\psi'\|^2_{L^2(\mathbb{R}^-)} - |q|^{-1}|\psi(0-)|^2. \tag{2.10}$$

*Analogously,*

$$Q_{0-} := \{\psi \in H^1(\mathbb{R}^+) \oplus H^1(\mathbb{R}^-), \psi(0-) = 0\} \tag{2.11}$$

*and the form reads*

$$B_{p,0}(\psi) = \|\psi'\|^2_{L^2(\mathbb{R}^+)} + \|\psi'\|^2_{L^2(\mathbb{R}^-)} + |p|^{-1}|\psi(0+)|^2. \tag{2.12}$$

*(3) For the Hamiltonian $H_U$, defined in (2.5), with $b = 0$ the energy space is*

$$Q_{\omega a} := \{\psi \in H^1(\mathbb{R}^+) \oplus H^1(\mathbb{R}^-),\ \psi(0+) = \omega a \psi(0-)\} \tag{2.13}$$

*and the form reads*

$$B_{\omega a}(\psi) = \|\psi'\|^2_{L^2(\mathbb{R}^+)} + \|\psi'\|^2_{L^2(\mathbb{R}^-)} + ac|\psi(0-)|^2. \tag{2.14}$$

*(4) For any other s.a.e. of $\widehat{H}_0$ the energy space is given by*

$$Q := H^1(\mathbb{R}^+) \oplus H^1(\mathbb{R}^-). \tag{2.15}$$

*To describe the action of the form we have to consider two cases:*

*(4a) if the Hamiltonian is of the type $H_U$ described in (2.5), with $b \neq 0$, then*

$$\begin{aligned} B_U(\psi) := {} & \|\psi'\|^2_{L^2(\mathbb{R}^+)} + \|\psi'\|^2_{L^2(\mathbb{R}^-)} + b^{-1}[d|\psi(0+)|^2 \\ & + a|\psi(0-)|^2 - 2\,\mathrm{Re}(\omega\overline{\psi(0+)}\psi(0-))] \end{aligned} \tag{2.16}$$

*(4b) if the Hamiltonian is of the type $H_{p,q}$ described in (2.6), with $p, q$ both different from zero, then*

$$B_{p,q}(\psi) := \|\psi'\|^2_{L^2(\mathbb{R}^+)} + \|\psi'\|^2_{L^2(\mathbb{R}^-)} + p^{-1}|\psi(0+)|^2 - q^{-1}|\psi(0-)|^2. \tag{2.17}$$

All energy spaces can be endowed with the structure of Hilbert space by introducing the scalar product

$$(\psi, \phi)_X = (\psi, \phi) + \lim_{\varepsilon\to 0+} \int_\varepsilon^{+\infty} \overline{\psi'(x)}\phi'(x)\,\mathrm{d}x + \lim_{\varepsilon\to 0+} \int_{-\infty}^{-\varepsilon} \overline{\psi'(x)}\phi'(x)\,\mathrm{d}x. \tag{2.18}$$

## 3. Global well-posedness in *D(H)*

**Theorem 3.1.** *Let $H$ be any self-adjoint extension of the operator $\widehat{H}_0$ defined in (2.3), (2.4). Let its domain be denoted by $D(H)$.*

*For any $\psi_0 \in D(H)$ equation (1.2) has a unique solution $\psi \in C(\mathbb{R}, D(H)) \cap C^1(\mathbb{R}, L^2(\mathbb{R}))$.*

*Furthermore, the following conservation laws hold at any time $t$ in the interval $[0, T)$:*

$$\|\psi(t)\| = \|\psi_0\| \tag{3.1}$$

$$\mathcal{E}[\psi(t)] = \mathcal{E}[\psi_0] \tag{3.2}$$

*where the energy functional is defined as*

$$\mathcal{E}[\psi] = \frac{1}{2}B(\psi) + \frac{\lambda}{4}\|\psi\|_4^4 \tag{3.3}$$

*and $B$ is the quadratic form associated with the operator $H$.*

**Proof.** First we prove local existence and blow-up alternative. Let us use the notation

$$\mathcal{X} = C^0([0, T], D(H)) \cap C^1([0, T], L^2(\mathbb{R}^2)) \tag{3.4}$$

and provide the space $\mathcal{X}$ with the norm

$$\|\psi\|_{\mathcal{X}} := \max_{t\in[0,T]} \|\psi(t)\|_H + \max_{t\in[0,T]} \|\partial_t\psi(t)\|. \tag{3.5}$$

Given $\psi_0 \in D(H)$, we define the function $G : \mathcal{X} \to \mathcal{X}$, as

$$G\psi := \mathrm{e}^{-\mathrm{i}H\cdot}\psi_0 - \mathrm{i}\lambda \int_0^{\cdot} \mathrm{d}s\, \mathrm{e}^{-\mathrm{i}(\cdot-s)H} |\psi(s)|^2 \psi(s). \tag{3.6}$$

First, it is immediately seen that

$$\|\mathrm{e}^{-\mathrm{i}Ht}\psi_0\|_H = \|\psi_0\|_H, \qquad \|\partial_t\, \mathrm{e}^{-\mathrm{i}Ht}\psi_0\| = \|H\psi_0\|. \tag{3.7}$$

Therefore,

$$\|\mathrm{e}^{-\mathrm{i}Ht}\psi_0\|_{\mathcal{X}} = \|\psi_0\|_H + \|H\psi_0\| \leqslant 2\|\psi_0\|_H. \tag{3.8}$$

Next, integrating by parts we obtain the identity

$$\begin{aligned}\int_0^t \mathrm{d}s\, \mathrm{e}^{-\mathrm{i}H(t-s)}|\psi(s)|^2\psi(s) &= -\mathrm{i}(H+m)^{-1}|\psi(t)|^2\psi(t) \\ &\quad + \mathrm{i}\mathrm{e}^{-\mathrm{i}Ht}(H+m)^{-1}|\psi_0|^2\psi_0 + m(H+m)^{-1}\int_0^t \mathrm{d}s\, \mathrm{e}^{-\mathrm{i}H(t-s)}|\psi(s)|^2\psi(s) \\ &\quad + 2\mathrm{i}(H+m)^{-1}\int_0^t \mathrm{d}s\, \mathrm{e}^{-\mathrm{i}H(t-s)}|\psi(s)|^2\partial_s\psi(s) \\ &\quad + \mathrm{i}(H+m)^{-1}\int_0^t \mathrm{d}s\, \mathrm{e}^{-\mathrm{i}H(t-s)}\psi^2(s)\partial_s\bar{\psi}(s).\end{aligned} \tag{3.9}$$

Owing to the integration by parts, standard estimates on the various terms yield

$$\|G\psi\|_{\mathcal{X}} \leqslant 2\|\psi_0\|_H + CT\|\psi\|_{\mathcal{X}}^3, \qquad \|G\psi - G\xi\|_{\mathcal{X}} \leqslant CT\big(\|\psi\|_{\mathcal{X}}^2 + \|\xi\|_{\mathcal{X}}^2\big)\|\psi - \xi\|_{\mathcal{X}}. \tag{3.10}$$

Let us fix $M := 4\|\psi_0\|_H$ and consider the ball of radius $M$ in the space $\mathcal{X}$, namely

$$\mathcal{Y} := \{\psi \in \mathcal{X}, \|\psi\|_{\mathcal{X}} \leqslant M\}. \tag{3.11}$$

From (3.10) one has

$$\|G\psi\|_{\mathcal{X}} \leqslant \frac{M}{2} + CTM^3, \qquad \|G\psi - G\xi\|_{\mathcal{X}} \leqslant CM^2T\|\psi - \xi\|_{\mathcal{X}}. \tag{3.12}$$

If one chooses $T = (2CM^2)^{-1}$, then $G$ is a contraction in $\mathcal{Y}$. By contraction lemma we immediately obtain the well-posedness of the problem (1.2) in $D(H)$. By standard techniques we know that there is a maximal time $T^\star(\psi_0)$ of existence for the solution. Since the size of the time interval chosen to construct the contraction depends on the $H$-norm of the solution only, and vanishes as such a norm diverges, either the solution is global or its $H$-norm diverges in finite time.

The conservation law for the $L^2$-norm is trivial. For the conservation of energy, note that

$$\partial_t(\psi(t), H\psi(t)) = -2\lambda\, \mathrm{Im}\,(|\psi(t)|^2\psi(t), H\psi(t)). \tag{3.13}$$

On the other hand,

$$\partial_t(\psi(t), |\psi(t)|^2\psi(t)) = \partial_t(\psi^2(t), \psi^2(t)) = 4\,\mathrm{Im}\,(|\psi(t)|^2\psi(t), H\psi(t)). \tag{3.14}$$

So conservation laws are proven.

Now we use the conservation laws to prove that strong solutions are global in time. We treat the case of $H = H_U$ with $b \neq 0$ (see definition (2.5)) only, being the other cases simpler. Suppose that

$T^\star(\psi_0)$ is finite. Integrating by parts,

$$\left\| \int_0^t \mathrm{d}s\, \mathrm{e}^{-\mathrm{i}H(t-s)} |\psi(s)|^2 \psi(s) \right\|_H \leqslant \|\psi(t)\|_\infty^2 \|\psi(t)\| + \|\psi_0\|_\infty^2 \|\psi_0\| + m \int_0^t \mathrm{d}s \|\psi(s)\|_\infty^2 \|\psi(s)\| + 3 \int_0^t \mathrm{d}s \|\psi(s)\|_\infty^2 \|H\psi(s)\| + 3|\lambda| \int_0^t \mathrm{d}s \|\psi(s)\|_\infty^2 \||\psi(s)|^2\psi(s)\|. \tag{3.15}$$

Using conservation of energy, conservation of $L^2$-norm, and inequality (4.6) first with $p = \infty$ and then with $p = 4$ one finds that, for any $t < T^\star(\psi_0)$,

$$\|\psi(t)\|_H \leqslant \|\psi_0\|_H + |\lambda| C \left[ (2 + t(m + 3|\lambda|C^2)) \|\psi_0\| + 3 \int_0^t \mathrm{d}s \|\psi(s)\|_H \right], \tag{3.16}$$

where $C$ is a constant depending on $\psi_0$ only. By a Gronwall-type estimate

$$\|\psi(t)\|_H \leqslant C(\psi_0)\, \mathrm{e}^{3|\lambda| K^2(\psi_0) T^\star(\psi_0)} < +\infty \tag{3.17}$$

so, by the blow-up alternative, $T^\star(\psi_0) = \infty$. The proof is complete. □

## 4. Global well-posedness in $Q$

In this section we prove the global well-posedness for the problem (1.2), provided that the operator $H$ is a s.a.e. of $\widehat{H}_0$ (see (2.3), (2.4)) with energy domain equal to $Q$ (see proposition 2.1). All results apply to the cases of $Q_0$, $Q_{0,+}$, $Q_{0-}$ and $Q_{\pm\omega}$.

First, we prove some estimates. Any element of $Q$ can be decomposed as

$$\psi(x) = \chi_+(x)\psi_+(x) + \chi_-(x)\psi_-(x), \tag{4.1}$$

so the following estimates hold:

$$\|\psi\|^2 = \tfrac{1}{2}\|\psi_+\|^2 + \tfrac{1}{2}\|\psi_-\|^2, \qquad \|\psi'\|^2 = \tfrac{1}{2}\|\psi_+'\|^2 + \tfrac{1}{2}\|\psi_-'\|^2, \qquad \|\psi_\pm'\| \leqslant \sqrt{2}\|\psi'\|, \tag{4.2}$$

$$|\psi(0+) - \psi(0-)| \leqslant \tfrac{1}{\sqrt{2}} \|\psi_+ - \psi_-\|_{H^1} \leqslant \tfrac{1}{\sqrt{2}} (\|\psi_+\|_{H^1} + \|\psi_-\|_{H^1}). \tag{4.3}$$

Besides, the norm introduced in (2.18) for elements of the energy space can be expressed as

$$\|\psi\|_Q^2 := \tfrac{1}{2}\|\psi_+\|_{H^1}^2 + \tfrac{1}{2}\|\psi_-\|_{H^1}^2 = \|\psi\|^2 + \|\psi'\|^2. \tag{4.4}$$

So, from the previous remarks

$$\|\psi\|_\infty \leqslant C\|\psi\|_Q, \|\psi'\| \leqslant C\|\psi\|_Q, \qquad |\psi(0+) - \psi(0-)| \leqslant C\|\psi\|_Q < \|\psi\|_\infty \leqslant C\|\psi\|_Q. \tag{4.5}$$

Finally, one-dimensional Gagliardo–Nirenberg's estimate can be extended to the space $Q$ as follows:

$$\|\psi\|_p \leqslant C_p \|\psi'\|^{\frac{1}{2}-\frac{1}{p}} \|\psi\|^{\frac{1}{2}+\frac{1}{p}}, \tag{4.6}$$

for any $p \in (2, +\infty]$.

The proof of the following lemma is easily obtained using (4.1).

**Lemma 4.1.** *For any function $\psi \in Q$,*

$$\||\psi|^2\psi\|_Q \leqslant C\|\psi\|_Q^3. \tag{4.7}$$

*Furthermore, for any couple of functions $\psi_1, \psi_2 \in Q$*

$$\||\psi_1|^2\psi_1 - |\psi_2|^2\psi_2\|_Q \leqslant C\big(\|\psi_1\|_Q^2 + \|\psi_2\|_Q^2\big)\|\psi_1 - \psi_2\|_Q. \tag{4.8}$$

**Lemma 4.2.** *Let $H$ be a s.a.e. of $\widehat{H}_0$ (see (2.3), (2.4)), and $X$ be the related energy space (see proposition 2.1). Then, for any $\psi_0 \in X, t \in \mathbb{R}$,*

$$\|\mathrm{e}^{-\mathrm{i}tH}\psi_0\|_X \leqslant \tilde{C}\|\psi_0\|_X \tag{4.9}$$

*where the constant $\tilde{C}$ depends on $\psi_0$ and $H$, but not on $t$.*

**Proof.** We treat the case at point 4 in proposition 2.1 only, the others being easier. Since $b \neq 0$, the domain of the form $B$ associated with the operator $H$ coincides with $Q$. Besides, the value of the form is conserved by the linear flow, namely $B(\psi_t) = B(\psi_0)$, where we used the notation $\psi_t = \mathrm{e}^{-\mathrm{i}tH}\psi_0$. Then, from the unitary character of the propagator $\mathrm{e}^{-\mathrm{i}tH}$ in $L^2(\mathbb{R})$,

$$|B(\psi_0)| \geqslant \|\psi_t'\|^2 - C_\infty |b|^{-1}(|a| + |d| + 1)\|\psi_t'\|\|\psi_0\| \tag{4.10}$$

where the constant $C_\infty$ was defined in (4.6).

From (4.10) one immediately has

$$\|\psi_t'\|^2 \leqslant C|B(\psi_0)| \tag{4.11}$$

where $C$ depends on $|a|$, $|b|$, $|d|$, and $\|\psi_0\|$ only. Conversely,

$$|B(\psi_0)| \leqslant \|\psi_0'\|^2 + |b|^{-1}(|a| + |d| + 1)\|\psi_0'\|\|\psi_0\| \leqslant C\|\psi_0\|_Q^2. \tag{4.12}$$

Then, from (4.12) and (4.10), one immediately has

$$\|\psi_t'\| \leqslant C\|\psi_0\|_Q,$$

and since $\|\psi_t\| = \|\psi_0\|$, the proof is complete. □

Now we can prove the well-posedness for the local Cauchy problem in the form domain of the s.a.e. $H$, provided that it coincides with the space $Q$.

**Theorem 4.3.** *Let $H$ be a s.a.e. of $\widehat{H}_0$ (see (2.3), (2.4)) whose corresponding quadratic form has domain $Q$. Then, for any $\psi_0 \in Q$, equation (1.2) has a unique solution $\psi \in C(\mathbb{R}, Q)$. Moreover, $L^2$-norm and energy (3.3) are conserved.*

**Proof.** Given $\psi_0 \in Q$ we consider the Banach space

$$\mathcal{Y} := \{\psi \in Q, \|\psi\|_Q \leqslant 2\tilde{C}\|\psi_0\|_Q\}$$

where $\tilde{C}$ is the constant appearing in (4.9). We define the operator $\Gamma$ acting on $L^\infty([0,T],\mathcal{Y})$, with $T$ to be specified:

$$(\Gamma\psi)(t) := \mathrm{e}^{-\mathrm{i}tH}\psi_0 - \mathrm{i}\lambda\int_0^t \mathrm{d}s\, \mathrm{e}^{-\mathrm{i}(t-s)H}|\psi(s)|^2\psi(s). \tag{4.13}$$

By lemmas 4.1 and 4.2

$$\|(\Gamma\psi)(t)\|_Q \leqslant \tilde{C}\|\psi_0\|_Q + CT\|\psi\|^3_{L^\infty([0,T],X)} \tag{4.14}$$

and for any $\psi_1, \psi_2 \in L^\infty([0,T], Q)$

$$\|(\Gamma\psi_1)(t) - (\Gamma\psi_2)(t)\|_Q \leqslant CT\left(\|\psi_1\|^2_{L^\infty([0,T],Q)} + \|\psi_2\|^2_{L^\infty([0,T],Q)}\right)\|\psi_1 - \psi_2\|_{L^\infty([0,T],Q)}. \tag{4.15}$$

From (4.14) and (4.15) it follows that for $T = \left(9C\tilde{C}^2\|\psi_0\|_Q^2\right)^{-1}$, $\Gamma$ is a contraction of $L^\infty([0,T],\mathcal{Y})$; then there exists a unique solution of equation (1.2) in $L^\infty([0,T],\mathcal{Y})$.

By a one-step bootstrap in (1.2) it is immediately seen that the solution actually belongs to $C^0([0,T],\mathcal{Y})$.

The time interval of local existence depends on the $Q$-norm of the solution only and vanishes as such a norm explodes, so one obtains the following alternative: either the solution is global in time, or its $Q$-norm diverges in finite time.

Conservation of the $L^2$-norm immediately follows from the obvious estimate

$$\partial_t \, \mathrm{e}^{\mathrm{i}Ht}\psi(t) = -\mathrm{i}\lambda \, \mathrm{e}^{\mathrm{i}Ht}|\psi(t)|^2\psi(t). \tag{4.16}$$

Furthermore, conservation of the energy is proven approximating $\psi_0$ by a sequence $\psi_{0,n} \in D(H)$, and comparing the associated global solutions with the aid of Gronwall's lemma. By the blow-up alternative, global existence follows and the theorem is proven. □

**Remark 4.4.** All proofs in this section can be extended to any s.a.e. $H$ of $\widehat{H}_0$ whose energy domain is $Q_0$, $Q_{0+}$, $Q_{0-}$ or $Q_{\pm\omega}$: the only condition is the stability of the energy domain under the action of the cubic nonlinearity.

## 5. Global well-posedness in $Q_{\omega a}$

In this section we prove the well-posedness for the problem (1.2) in the form domain $Q_{\omega a}$ of a hamiltonian operator $H$ defined by boundary conditions (2.5) with $b = 0$.

As already pointed out, this case is the most delicate. To treat it, we must prove a generalized version of the 'integrated by parts' form of the Duhamel formula. To that purpose we need to extend the action of the resolvent of $H$ from $L^2(\mathbb{R})$ to the whole space $Q^\star$.

### *5.1. Dual of energy spaces*

By standard argument of functional analysis (see e.g. [28], section 4.5), one proves

**Proposition 5.1.** *The spaces $Q_0^\star$, $Q_{\omega a}^\star$ and $Q^\star$ can be represented as follows:*

$$Q_0^\star = \left\{ f \in H^{-1}(\mathbb{R}), \text{ s.t.} \int_{\mathbb{R}} \frac{\widehat{f}(k)}{k^2+1} \, \mathrm{d}k = 0 \right\} \tag{5.1}$$

$$Q_{\omega a}^\star = Q_0^\star \oplus \mathrm{Span}(\delta^{\omega a}(0+)) \tag{5.2}$$

$$Q^\star = Q_0^\star \oplus \mathrm{Span}(\delta(0+), \delta(0-)). \tag{5.3}$$

*We used the notation $\delta(0\pm)$ to denote the functionals acting as follows:*

$$\langle \delta(0\pm), \psi \rangle_{X^\star, X} = \lim_{x \to 0\pm} \psi(x).$$

*Besides, we denoted by $\delta^{\omega a}(0+)$ the functional that vanishes on $Q_0 \oplus \mathrm{Span}(\varphi_{-\omega a^{-1}})$ and gives $\omega a$ when acting on $\varphi_{\omega a}$.*

**Corollary 5.2** (Decomposition of dual spaces)**.** *For any $f \in Q^\star$ there exist $f_0 \in Q_0^\star$, $f_{\omega a} \in Q_{\omega a}^\star$, $\alpha, \beta \in \mathbb{C}$, such that*

$$f = f_0 + \alpha\delta(0+) + \beta\delta(0-) \tag{5.4}$$

$$f = f_{\omega a} + \frac{\beta - \omega a^{-1}\alpha}{1+a^2}[a^2\delta(0-) - \overline{\omega} a \delta(0+)]. \tag{5.5}$$

**Proof.** Let $f$ be an element of $Q^\star$. For any $\psi \in Q$ define

$$f_0(\psi) := f(\psi_0) \tag{5.6}$$

where $\psi_0$ is the $H_0^1$-component of $\psi$. Moreover, define

$$\alpha := f(\varphi_+), \qquad \beta := f(\varphi_-) \tag{5.7}$$

and decomposition (5.4) is proven. Furthermore, we define the functional $f_{\omega a}$ as follows:

$$f_{\omega a} := f_0(\psi) + \frac{\omega a\alpha + \beta}{1 + a^2}[\overline{\omega}a\delta(0+) + \delta(0-)]. \tag{5.8}$$

To see that $f_{\omega a}$ is indeed in $Q^\star_{\omega a}$ it is sufficient to show that it is orthogonal to $Q_{-\omega a^{-1}}$. But, since $\dim Q\backslash Q_{\omega a} = 1$, the space orthogonal to $Q_{-\omega a^{-1}}$ must be one dimensional, so it must be spanned by the function $\varphi_{-\omega a^{-1}}$, and (5.5) is proven. □

**Definition 5.3.** *For any energy domain $X$ we define the norm in $X^\star$ by*

$$\|f\|_{X^\star} := \sup_{\psi\in X\backslash\{0\}} \frac{|\langle f, \psi\rangle_{X^\star,X}|}{\|\psi\|_X}. \tag{5.9}$$

**Remark 5.4.** It appears that, for any $f \in X^\star$, $\omega \in \mathbb{C}$ with $|\omega| = 1$, and $a \in \mathbb{R}$,

$$\|f_0\|_{Q_0^\star} \leqslant \|f_{\omega a}\|_{Q^\star_{\omega a}} \leqslant \|f\|_{Q^\star}. \tag{5.10}$$

Vice versa, given $f \in Q_0^\star$ it is possible to define its *trivial extension* $\tilde{f}$ to $Q_{\omega a}$ as the functional acting like $f$ on $Q_0$ and vanishing on $\varphi_{\omega a}$. Obviously, it can be further extended to $\tilde{\tilde{f}}$ that vanishes on $\varphi_{-\omega a^{-1}}$. One has

$$\|f\|_{Q_0^\star} = \|\tilde{f}\|_{Q^\star_{\omega a}} = \|\tilde{\tilde{f}}\|_{Q^\star}. \tag{5.11}$$

### *5.2. Extension of the resolvent to $Q^\star$*

We start by extending the resolvent of the free Laplacian to $Q^\star$.

**Definition 5.5.** *Let $H_0$ be the s.a.e. of $\widehat{H}_0$ (see (2.3), (2.4)), defined on $H^2(\mathbb{R})$. For any $z \in \mathbb{C}\backslash(-\infty, 0]$ denote by $R_0(z)$ the resolvent operator $(H_0 + z)^{-1}$, acting on $L^2(\mathbb{R})$.*

*We define the extended free resolvent $\widetilde{R}_0(z)$ as follows. Given $f \in Q^\star$,*

$$\widetilde{R}_0(z) f := \mathcal{F}^{-1}\frac{\widehat{f}_1(k)}{k^2 + z} \tag{5.12}$$

*where, according to (5.4), (5.5), $f_1$ is the $H^{-1}$-component of $f$, $\widehat{f}_1$ is its Fourier transform as a Schwartz distribution and $\mathcal{F}^{-1}$ denotes the inverse Fourier transform in the same space.*

We point out the following:

- for any $f \in Q^\star$, $\widetilde{R}_0(z) f \in H^1(\mathbb{R})$;
- $\widetilde{R}_0(z)$ is not invertible, indeed its kernel coincides with the subspace of $Q^\star$ generated by $\delta(0+) - \delta(0-)$. However, its restriction on $H^{-1}(\mathbb{R})$ is invertible;
- $\widetilde{R}_0(z)$ is bounded as an operator from $Q^\star$ to $H^1(\mathbb{R})$. Indeed,

$$\|\widetilde{R}_0(z) f\|^2_{H^1(\mathbb{R})} \leqslant C\int_{\mathbb{R}} \frac{|\widehat{f}_1(k)|^2}{k^2 + 1}\,\mathrm{d}k \leqslant C\|f\|^2_{Q^\star}. \tag{5.13}$$

As a second step we extend to $Q^\star$ the resolvent of the s.a.e. of $\widehat{H}_0$ with Dirichlet boundary condition at zero.

**Definition 5.6.** *Let $H_{0,0}$ be the s.a.e. of $\widehat{H}_0$ (see (2.1), (2.2)) whose domain $D_D$ contains all functions in $H^2(\mathbb{R})$ vanishing at $x = 0$. For any $z \in \mathbb{C}\backslash(-\infty, 0]$ denote by $R_D(z)$ the resolvent operator $(H_{0,0} + z)^{-1}$ acting on $L^2(\mathbb{R})$.*

*We define the action of the extended resolvent $\widetilde{R}_D(z)$ on $f \in Q^\star$ as*

$$\widetilde{R}_D(z) f := \widetilde{R}_0(z) f - \frac{\varphi^z}{2\sqrt{z}}\langle f, \varphi^z\rangle_{Q^\star,Q} \tag{5.14}$$

*where the functions $\varphi^z$ were introduced in section 2.1.*

Note that

- $\widetilde{R}_D(z)f = R_D(z)f$ if $f \in L^2(\mathbb{R})$.
- $\widetilde{R}_D(z)$ is a continuous linear map from $Q^\star$ to $H_0^1(\mathbb{R})$.
  Indeed, from continuity of $\widetilde{R}_0(z)$ and continuity of the second term in the definition (5.14) of $\widetilde{R}_0(z)$, we have that $\widetilde{R}_D(z)$ is continuous from $Q^\star$ to $H^1(\mathbb{R})$. Now, fix $f \in Q^\star$ and consider a sequence $f_n \in L^2(\mathbb{R}) \cap Q^\star$ that converges to $f$ in the topology of $Q^\star$. Obviously, $\widetilde{R}_D f_n(0) = 0$, and by continuity $\widetilde{R}_D f_n$ converges to $\widetilde{R}_D f$ in $H^1(\mathbb{R})$. But this implies pointwise convergence, then $\widetilde{R}_D f(0) = 0$.
- Decomposing $f$ as in (5.4) one immediately has

$$\widetilde{R}_D(z)f = \widetilde{R}_0(z)f_0 - \frac{\varphi^z}{2\sqrt{z}}\langle f_0, \varphi^z\rangle_{Q^\star,Q}. \tag{5.15}$$

- From (5.15),

$$\widetilde{R}_D(z)f = \widetilde{R}_D(z)f_0.$$

Now we can extend to $Q^\star$ the resolvent of any s.a.e. of $\widehat{H}_0$.

**Definition 5.7.** *Let $H$ be any s.a.e. of $\widehat{H}_0$. For any $z$ in the resolvent set of $H$ we define the extended resolvent $\widetilde{R}(z)$ as follows:*

$$\widetilde{R}(z)f := \widetilde{R}_D(z)f + \sum_{j,k=\pm}\frac{\mu_{j,k}(z)}{2\sqrt{z}}\varphi_j^z\big\langle f, \varphi_k^z\big\rangle_{Q^\star,Q} \tag{5.16}$$

*where the coefficients $\alpha$ and $\beta$ were defined in (5.4), the function $\varphi_j^z$ in (2.2) and the parameters $\mu_{j,k}(z)$ give the difference between the ordinary resolvent operators $R(z) - R_D(z)$ according to Krein's formula (see [4], chapter 7, section 84).*

As in the previously discussed cases,

- $\widetilde{R}(z)f = (H+z)^{-1}f$ if $f \in L^2(\mathbb{R})$;
- it is easily seen that $\widetilde{R}(z)$ is a continuous linear map from $Q^\star$ to $Q$;
- by decomposition (5.4), one has

$$\begin{aligned}\widetilde{R}(z)f := \widetilde{R}_D(z)f_0 &+ \sum_{j,k=\pm}\frac{\mu_{j,k}(z)}{2\sqrt{z}}\varphi_j^z\big\langle f_0, \varphi_k^z - \varphi_k\big\rangle_{Q^\star,Q}\\ &+ \frac{\alpha}{2\sqrt{z}}\sum_{j=\pm}\mu_{j,+}(z)\varphi_j^z + \frac{\beta}{2\sqrt{z}}\sum_{j=\pm}\mu_{j,-}(z)\varphi_j^z.\end{aligned} \tag{5.17}$$

**Remark 5.8.** Applying Krein's theory one easily verifies that if $H$ is a hamiltonian operator defined by the boundary condition (2.5) with $b = 0$, $a \neq \pm 1$, then the action of its extended resolvent is represented by the integral kernel

$$\begin{aligned}\widetilde{R}_{\omega,a,c}(z; x, y)f := \widetilde{R}_D(z; x, y)f &+ \frac{1}{(a^2+1)\sqrt{z}+ac}\big[a^2\theta_+(x)\theta_+(y)\,\mathrm{e}^{-\sqrt{z}x}\,\mathrm{e}^{-\sqrt{z}y}\\ &+\omega a\theta_+(x)\theta_-(y)\,\mathrm{e}^{-\sqrt{z}x}\,\mathrm{e}^{\sqrt{z}y} + \overline{\omega}a\theta_-(x)\theta_+(y)\,\mathrm{e}^{\sqrt{z}x}\,\mathrm{e}^{-\sqrt{z}y}\\ &+\theta_-(x)\theta_-(y)\,\mathrm{e}^{\sqrt{z}x}\,\mathrm{e}^{\sqrt{z}y}\big]\end{aligned} \tag{5.18}$$

where the functions $\varphi_\pm^z$ were explicitly written.

Moreover,

- for any $f \in Q^\star$, we find $\widetilde{R}_{\omega,a,c}(z) f \in Q_{\omega a}$. To prove it, we just note that for any $f \in Q^\star$

$$\begin{aligned}\widetilde{R}_{\omega,a,c}(z) f(0+) &= \frac{1}{(a^2+1)\sqrt{z}+ac}[a^2\langle f, \varphi_+^z\rangle_{Q^\star,Q} + \omega a\langle f, \varphi_-^z\rangle_{Q^\star,Q}]\\ \widetilde{R}_{\omega,a,c}(z) f(0-) &= \frac{1}{(a^2+1)\sqrt{z}+ac}[\overline{\omega}a\langle f, \varphi_+^z\rangle_{Q^\star,Q} + \langle f, \varphi_-^z\rangle_{Q^\star,Q}].\end{aligned} \tag{5.19}$$

  Then the correct boundary condition is fulfilled.
- $\widetilde{R}_{\omega,a,c}(z)$ is not invertible, since its kernel coincides with the subspace of $Q^\star$ generated by $\delta(0+) - \omega a\delta(0-)$. However, its restriction to $Q^\star_{\omega a}$ is invertible.
- $\widetilde{R}_{\omega,a,c}(z)$ is bounded as an operator from $Q^\star$ to $Q_{\omega a}$.

### 5.3. *Proof of the well-posedeness*

Before proving the well-posedness for the problem (1.2) in the space $Q^\star_{\omega a}$, we note that by using continuity of $\widetilde{R}(m)$ one immediately has the following lemma:

**Lemma 5.9.** *Let $f$ be a map in $C^0([0,T), Q_\gamma) \cap C^1([0,T), Q^\star_\rho)$, with $\gamma, \rho \in \mathbb{C}$. Then*

$$\partial_t\, \mathrm{e}^{\mathrm{i}(H+m)t}\widetilde{R}(m) f(t) = \mathrm{i}\mathrm{e}^{\mathrm{i}(H+m)t} f(t) + \mathrm{e}^{\mathrm{i}(H+m)t}\widetilde{R}(m)\partial_t f(t), \tag{5.20}$$

*where $H$ is any s.a.e. of $\widehat{H}_0$ and the derivative of $f$ is to be understood in $Q^\star_\rho$.*

Then formula (3.9) can be generalized to

$$\begin{aligned}\int_0^t \mathrm{d}s\, \mathrm{e}^{-\mathrm{i}H(t-s)}|\psi(s)|^2\psi(s) &= -\mathrm{i}\widetilde{R}(m)|\psi(t)|^2\psi(t) + \mathrm{i}\mathrm{e}^{-\mathrm{i}Ht}\widetilde{R}(m)|\psi_0|^2\psi_0\\ &\quad + m\widetilde{R}(m)\int_0^t \mathrm{d}s\, \mathrm{e}^{-\mathrm{i}H(t-s)}|\psi(s)|^2\psi(s) + 2\mathrm{i}\widetilde{R}(m)\int_0^t \mathrm{d}s\, \mathrm{e}^{-\mathrm{i}H(t-s)}|\psi(s)|^2\partial_s\psi(s)\\ &\quad + \mathrm{i}\widetilde{R}(m)\int_0^t \mathrm{d}s\, \mathrm{e}^{-\mathrm{i}H(t-s)}\psi^2(s)\partial_s\bar{\psi}(s).\end{aligned} \tag{5.21}$$

We finally prove global well-posedness in $Q_{\omega a}$.

**Theorem 5.10** (Existence and uniqueness for global solutions in $\boldsymbol{Q_{\omega a}}$)**.** *Let $H$ be any self-adjoint extension of the operator $\widehat{H}_0$ (see (2.3), (2.4)), defined by the boundary conditions (2.5) with $b = 0$, $a \neq \pm 1$. Let its energy domain be denoted by $Q_{\omega a}$.*

*Then for any $\psi_0 \in Q_{\omega a}$ equation (1.2) has a unique solution $\psi \in C([0,+\infty), Q_{\omega a}) \cap C^1([0,+\infty), Q^\star_{\omega a})$. Moreover, for the solution the conservation laws of $L^2$-norm and of energy $B_{\omega a}(\psi(t)) + \lambda/2\|\psi(t)\|_4^4$ hold.*

**Proof.** We define $\mathcal{Z} := C([0,T), Q_{\omega a}) \cap C^1([0,T), Q^\star_{\omega a})$ and show that the map

$$\Theta : \mathcal{Z}_r \to \mathcal{Z}_r \qquad \psi \mapsto \mathrm{e}^{-\mathrm{i}H\cdot}\psi_0 - \mathrm{i}\lambda\int_0^\cdot \mathrm{d}s\, \mathrm{e}^{-\mathrm{i}H(\cdot-s)}|\psi(s)|^2\psi(s), \tag{5.22}$$

where $\mathcal{Z}_r$ is a closed ball of radius $r$ (to be chosen) in $\mathcal{Z}$, is a contraction, for small $T$.

The proof of the first estimate of interest closely follows the line of the proof of the bounds in theorem 3.1. Some more care is required in the estimate of the two last terms of (5.21). We show how to proceed considering the second-last term.

By continuity of $\widetilde{R}(m) : Q^\star_{\omega a^{-1}} \to Q_{\omega a}$ one gets

$$\|\widetilde{R}(m)|\psi(t)|^2\partial_t\psi(t)\|_{Q_{\omega a}} \leqslant C\||\psi(t)|^2\partial_t\psi(t)\|_{Q^\star_{\omega a^{-1}}}. \tag{5.23}$$

Furthermore, approximating $\partial_t\psi(t)$ by a sequence in $L^2$ and using continuity it is clear that the multiplication by $|\psi(t)|^2$ is to be understood by duality; therefore,

$$\langle|\psi(t)|^2\partial_t\psi(t),\xi\rangle_{Q^\star_{\omega a^{-1}},Q_{\omega a^{-1}}} = \langle\partial_t\psi(t),|\psi(t)|^2\xi\rangle_{Q^\star_{\omega a},Q_{\omega a}}. \tag{5.24}$$

where $\xi$ is an element of $Q_{\omega a^{-1}}$. Hence,

$$\||\psi(t)|^2\partial_t\psi(t)\|_{Q^\star_{\omega a^{-1}}} \leqslant \|\psi(t)\|^2_\infty\|\partial_t\psi(t)\|_{Q^\star_{\omega a}} \leqslant C\|\psi(t)\|^2_{Q_{\omega a}}\|\partial_t\psi(t)\|_{Q^\star_{\omega a}}. \tag{5.25}$$

Then

$$\|\widetilde{R}(m)|\psi(t)|^2\partial_t\psi(t)\|_{Q_{\omega a}} \leqslant C\|\psi(t)\|^2_{Q_{\omega a}}\|\partial_t\psi(t)\|_{Q^\star_{\omega a}}. \tag{5.26}$$

Now we estimate the time derivative of $\Theta\psi(t)$ as a functional on $Q^\star_{\omega a}$. First, we define it in the usual way: given $\zeta\in Q_{\omega a}$

$$\langle\Theta\psi(t),\zeta\rangle_{Q^\star_{\omega a},Q_{\omega a}} := (\Theta\psi(t),\zeta).$$

Then

$$\begin{aligned}\langle\partial_t\Theta\psi(t),\zeta\rangle_{Q^\star_{\omega a},Q_{\omega a}} &:= \partial_t(\Theta\psi(t),\zeta) = \mathrm{i}\lambda(|\psi(t)|^2\psi(t),\zeta) + B_{\omega a}(\psi(t),\zeta)\\ &\leqslant C\|\psi(t)\|^3_{Q_{\omega a}}\|\zeta\|_{Q_{\omega a}}\end{aligned} \tag{5.27}$$

where we exploited formula (4.16), with $\mathrm{e}^{\mathrm{i}Ht}\Theta\psi(t)$ replacing $\mathrm{e}^{\mathrm{i}Ht}\psi(t)$ in the lhs. The proof proceeds as in theorem 3.1, so we can conclude

$$\|\Theta\psi\|_{\mathcal{Z}} \leqslant \tilde{C}\|\psi_0\|_{Q_{\omega a}} + CT\|\psi\|_{\mathcal{Z}}. \tag{5.28}$$

For the proof of the Lipschitz condition we first consider the $C([0,T),Q_{\omega a})$ norm. The situation is analogous to the one discussed in the proof of the corresponding point in theorem (3.1). We show how to proceed for estimating the most complicated term:

$$\begin{aligned}\||\psi(t)|^2\partial_t\psi(t) - |\xi(t)|^2\partial_t\xi(t)\|_{Q^\star_{\omega a^{-1}}} &\leqslant \||\psi(t)|^2 - |\xi(t)|^2\|_Q\|\partial_t\psi(t)\|_{Q^\star_{\omega a}}\\ &\quad + \|\xi(t)\|^2_{Q_{\omega a}}\|\partial_t\psi(t) - \partial\xi(t)\|_{Q^\star_{\omega a}}\end{aligned}$$

and then

$$\begin{aligned}&\|\Theta\psi - \Theta\xi\|_{C([0,T),Q_{\omega a})}\\ &\quad = |\lambda|\left\|\widetilde{R}(m)\int_0^\cdot \mathrm{d}s\,\mathrm{e}^{-\mathrm{i}H(\cdot-s)}\left(|\psi(s)|^2\partial_s\psi(s) - |\xi(s)|^2\partial_s\xi(s)\right)\right\|_{C([0,T),Q_{\omega a})}\\ &\quad \leqslant CT\left(\|\psi\|^2_{\mathcal{Z}} + \|\xi\|^2_{\mathcal{Z}}\right)\|\psi - \xi\|_{\mathcal{Z}}.\end{aligned} \tag{5.29}$$

The Lipschitz bound in the norm $C^1([0,T),\mathcal{Z})$ is easily obtained applying formula (5.27). So we have

$$\|\Theta\psi - \Theta\xi\|_{\mathcal{Z}} \leqslant CT\left(\|\psi\|^2_{\mathcal{Z}} + \|\xi\|^2_{\mathcal{Z}}\right)\|\psi - \xi\|_{\mathcal{Z}}. \tag{5.30}$$

Mimicking the proof of theorem 3.1 from formula (3.10) to the end we prove that $\Theta$ is a contraction when restricted to a suitable ball centered at the origin and a suitable time interval $[0,T/2]$. Since both the size of the ball and $T$ depend on $\|\psi_0\|_{Q_{\omega a}}$ only, we have the blow-up alternative.

Conservation of the $L^2$-norm and of the energy, and therefore the global character of the solution, can be proved following the line used for the analogous issues for solutions in $Q$. The only delicate point arises when proving the conservation of the energy. Indeed, the persistence of a boundary condition in the definition of the energy domain prevents one from directly extending the result. We sketch the necessary modifications to the proof of theorem 4.3: first, as in the case of $Q$, one approximates the initial data $\psi_0$ by a sequence $\psi_{0,n}$ in $D(H)$; second, denoted by $\psi_n$ the solution to (1.2) with initial data $\psi_{0,n}$, and using that $\psi(t)$ lies in $Q_{\omega a}$, one proves that

$$\|\psi(t) - \psi_n(t)\| \leqslant C\,\mathrm{e}^{C_T T}\|\psi_0 - \psi_{0,n}\|, \qquad \forall\, t \in [0, T); \tag{5.31}$$

for any $T$ in the existence interval of $\psi$. Therefore, $\psi_n(t)$ converges to $\psi(t)$ uniformly in $L^2(\mathbb{R})$; third, integrating by parts in Duhamel's formula (see (5.21)) one gets

$$\begin{aligned}\|\psi_n(t) - \psi(t)\|_{Q_{\omega a}} &\leqslant C\left(\|\psi_{0,n} - \psi_0\|_{Q_{\omega a}} + \max_{t\in[0,T]}\|\psi_n(t) - \psi(t)\|\right)\\ &\quad + C\int_0^t \mathrm{d}s\,\|\partial_s\psi_n(s) - \partial_s\psi(s)\|_{Q^\star_{\omega a}}.\end{aligned} \tag{5.32}$$

Finally, using (5.31) and Gronwall's inequality again one has that $\psi_n(t)$ converges to $\psi(t)$ in $Q_{\omega a}$. Then, by continuity of energy in $Q_{\omega a}$, the proof is complete. □

## 6. Well-posedness in $L^2(\mathbb{R})$

The theory of well-posedness in $L^2(\mathbb{R})$ is analogous to the theory for the free NLS in one dimension as exposed, for example, in [12]. However, some additional estimates and details are needed, if the operator $H$ has a non-trivial point spectrum. We stress that this case is not exceptional: sufficient conditions are given in [5], formula (2.13), for the case in (2.5), whereas for hamiltonian operators as in (2.6) the pure point spectrum is non-trivial if and only if either $p$ or $q$ is negative.

It turns out that the presence of a point spectrum prevents us from extending Strichartz's estimate to an infinite time interval.

As usual, we call *admissible pair* any couple of real numbers $(q, r)$ such that

$$\frac{2}{q} = \frac{1}{2} - \frac{1}{r}.$$

Let us denote by $H$ a s.a. operator satisfying (2.5) or (2.6) and by $U(\cdot)$ the corresponding unitary group.

**Lemma 6.1** (Strichartz estimates)**.** *Let $(q, r)$ be an admissible pair. For every $\psi_0 \in L^2(\mathbb{R})$ the following estimates hold true:*

*(i)*
$$\|U(\cdot)\psi_0\|_{L^q((0,T),L^r(\mathbb{R}))} \leqslant C\|\psi_0\|; \tag{6.1}$$

*(ii) given any $t_0 \in (0, T)$, $(\gamma, \rho)$ an admissible pair and $f \in L^{\gamma'}((0, T), L^{\rho'}(\mathbb{R}))$, the function*

$$\Phi(t) = \int_{t_0}^t U(t - s)f(s)\,\mathrm{d}s$$

*belongs to $L^q((0, T), L^r(\mathbb{R})) \cap C((0, T), L^2(\mathbb{R}))$. Moreover,*

$$\|\Phi\|_{L^q((0,T),L^r(\mathbb{R}))} \leqslant C\|f\|_{L^{\gamma'}((0,T),L^{\rho'}(\mathbb{R}))}. \tag{6.2}$$

**Proof.** Theorem 3.1 in [5] (formulas (3.5) and (3.6)) gives an explicit characterization of the propagator of generalized point interactions, separating the contributions of absolutely continuous and pure point spectrum (their singular spectrum is empty). It follows from the explicit form of the decomposition $U(t) = U_{ac}(t) + U_{pp}(t)$ that the following dispersive estimate holds:

$$\|U_{ac}(t)\psi_0\|_{L^\infty(\mathbb{R})} \leqslant C|t|^{-\frac{1}{2}}\|\psi_0\|_{L^1(\mathbb{R})}.$$

So, the standard proof of Strichartz estimates (see for example theorem 2.3.3 in [12]) applies without difficulties taking into account the regularizing properties of the a.c. part of the resolvent of $H$; therefore, Strichartz estimates (6.1) and (6.2) are proven with $U_{ac}(t)$ in the place of $U(t)$. Let us consider the contribution of $U_{pp}(t)$. We have

$$U_{pp}(t)\psi_0 = \sum_{j=1}^{\#\{E_j\}} \mathrm{e}^{-\mathrm{i}E_j t}(\phi_j, \psi_0)\phi_j \tag{6.3}$$

where $\{E_j\}$ is the set of the eigenvalues of $H$, $\#\{E_j\}$ denotes its cardinality and $\phi_j$'s are the corresponding eigenvectors. Note that the latter belong to every space $L^p(\mathbb{R})$. This yields the following estimate:

$$\|U_{pp}\psi_0\|_{L^q((0,T),L^r(\mathbb{R}))} \leqslant \left(\sum_{j=1}^{\#\{E_j\}} \|\phi_j\|\|\phi_j\|_r\right) T^{\frac{1}{q}}\|\psi_0\| \tag{6.4}$$

and (6.1) is proven.

Now, for the proof of (ii), let us write $\Phi(t) = \Phi_{ac}(t) + \Phi_{pp}(t)$ with obvious meaning of the notation. We get

$$\|\Phi_{pp}\|_{L^q((0,T),L^r(\mathbb{R}))} \leqslant \left(\sum_{j=1}^{\#\{E_j\}} \|\phi_j\|_\rho\|\phi_j\|_r\right) T^{\frac{1}{\gamma}+\frac{1}{q}}\|f\|_{L^{\gamma'}((0,T),L^{\rho'}(\mathbb{R}))}. \tag{6.5}$$

Finally, a direct estimate shows that $\Phi_{pp} \in C((0,T), L^2(\mathbb{R}))$. This concludes the proof of the lemma. □

Now we come to the well-posedness of dynamics in $L^2(\mathbb{R})$.

**Theorem 6.2.** *For any $\psi_0 \in L^2(\mathbb{R})$ there exists a unique solution $\psi \in C(\mathbb{R}, L^2(\mathbb{R})) \cap L^8_{\mathrm{loc}}(\mathbb{R}, L^4(\mathbb{R}))$ of problem (1.2). Moreover, $L^2$-norm is conserved.*

**Proof.** The proof strictly follows the proof of theorem 4.6.1. in [12], for the particular case $\alpha = 2$, $p = 4$, $q = 8$. The proof of the conservation law only (step 5 in Cazenave's proof) requires a modification: the regularizing operator $J_\varepsilon$ must be defined by

$$J_\varepsilon := (I + \varepsilon H)^{-1}. \tag{6.6}$$

Note that for $\varepsilon$ sufficiently small $J_\varepsilon$ is well defined. We need to prove the following facts:

(1) As $\varepsilon$ goes to zero, $J_\varepsilon \to I$ strongly in $L^p(\mathbb{R})$, for any $1 < p < \infty$.
(2) As $\varepsilon$ goes to zero, $J_\varepsilon \to I$ strongly in $L^q_{\mathrm{loc}}(\mathbb{R}, L^p(\mathbb{R}))$, where $(q, p)$ is an admissible pair.
(3) $J_\varepsilon$ is bounded as an operator in $L^p$, uniformly in $\varepsilon$.

In order to prove (1), (2) and (3), it is convenient to use Krein's formula (see [4], chapter 7, section 84). Given $f \in L^p(\mathbb{R})$ one obtains

$$J_\varepsilon f := I_\varepsilon f + \varepsilon^{-1} \sum_{j,k=\pm} \frac{p_{j,k}(\varepsilon^{-1})}{2\sqrt{\varepsilon^{-1}}} \varphi_j^{\varepsilon^{-1}} \left\langle \varphi_k^{\varepsilon^{-1}}, f\right\rangle_{L^{p'}(\mathbb{R}),L^p(\mathbb{R})} \tag{6.7}$$

where $I_\varepsilon$ denotes the operator $\left(I - \varepsilon\partial_x^2\right)^{-1}$. From formula (12) in the cited section of [4] it appears that $p_{j,k}(\varepsilon^{-1})$ is bounded for $\varepsilon \to 0$. Furthermore, from proposition 1.5.2 in [12] it is clear that 1, 2 and 3 hold for $I_\varepsilon$. It remains to show that $\varepsilon^{-1/2}\varphi_j^{\varepsilon^{-1}}\left\langle f, \varphi_k^{\varepsilon^{-1}}\right\rangle_{L^{p'}(\mathbb{R}),L^p(\mathbb{R})}$ vanishes in $L^p(\mathbb{R})$. Indeed, it is clear that $I_\varepsilon\chi_+ f \to f$; therefore,

$$\varepsilon^{-1/2}\varphi_j^{\varepsilon^{-1}}\left\langle f, \varphi_k^{\varepsilon^{-1}}\right\rangle_{L^{p'}(\mathbb{R}),L^p(\mathbb{R})} = \chi_- I_\varepsilon \chi_+ f \to 0$$

in $L^p(\mathbb{R})$. This proves 1. Conditions 2 and 3 can be proven in the same way. The rest of the proof follows exactly the line of [12], except that $H^1(\mathbb{R})$ has to be replaced by the energy space $X$ associated with the operator $H$. □

## 7. Extensions and perspectives

### *7.1. Finitely many-point interactions*

Our results immediately extend to the case of finitely many-(say $n$) point interactions. Imagine that the interactions are located at $y_1 < \cdots < y_n$, which we collectively denote by the $n$-dimensional vector $Y$. The linear part of the dynamics is then generated by a s.a. extension $H$ of the free Laplacian defined on $C_0^\infty\big(\mathbb{R}\backslash\big(\cup_{j=1}^n \{y_j\}\big)\big)$.

By the von Neumann's theory for s.a. extensions, any such $H$ is characterized by linear conditions on the quantities $\psi(y_j)$, $\psi'(y_j)$ that generalize and mix (2.5), (2.6). A generalization of proposition (2.1) holds, but we do not give it in full generality. We just point out that all form domains are included in

$$Q^Y = H^1(-\infty, y_1) \oplus H^1(y_n, +\infty) \oplus_{i=1}^{n-1} H^1(y_i, y_{i+1})$$

and include

$$Q_0^Y = \{\psi \in Q^Y, \psi(y_j) = 0, j = 1, \dots, n\}.$$

Similarly, all duals of form domains include

$$Q_0^{Y,\star} = \left\{ f \in H^{-1}(\mathbb{R}), \text{ s.t.} \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}k y_j} \frac{\widehat{f}(k)}{k^2+1}\,\mathrm{d}k = 0 \right\}$$

and are included in

$$Q^{Y,\star} = Q_0^\star \oplus_{j=1}^n \mathrm{Span}(\delta(y_j\pm)).$$

Estimates in section 4 still hold and section 3 (including well-posedness in the operator domain, conservation laws, global character of the solutions) can be rewritten without modification.

Section 4 is valid for this case too, except that proofs must be modified, since the expressions involving the boundary conditions are much more complicated.

The procedure and results in section 5 can be extended to this general case too, but writing the resolvents in all details can be quite cumbersome.

Finally, proving the global well-posedness of the problem in $L^2(\mathbb{R})$ is not immediate. The main hindrance consists in proving Strichartz's estimates. More specifically, due to the possible presence of bound states with positive energy (see e.g. [6]), it is not immediate to treat separately the discrete and the continuous part of the spectrum.

### *7.2. More general nonlinearity power*

All results on local well-posedness can be immediately extended to a problem analogous to (1.2), but with the nonlinear term replaced by $\lambda|\psi|^p\psi$. Global well-posedness holds in the energy domain if $p < 4$ or $\lambda > 0$. As in the standard NLS, for attractive nonlinearity ($\lambda < 0$) and $p \geqslant 4$, blow-up phenomena can occur: it can be proven by ordinary methods ([12]).

Nevertheless, it would be interesting to understand what happens when the blow-up occurs by concentration of the wave packet at the point where the interaction is located. If the interaction is very singular (e.g. $\delta'$), the blow-up profile is expected to be non-standard.

### *7.3. Nonlinear boundary condition*

A more general problem can be considered defining nonlinear boundary conditions, i.e. considering the coefficients $a, b, c, d, \omega$ in (2.5) and $p, q$ in (2.6) as functions of the solution $\psi(t)$, following the line of [2, 3].

From the point of view of the modeling, such a system would describe nonlinear phenomena that originate outside the system and are localized in the defect. Concerning the mathematical investigation, it is clear that the methods would be considerably different, due to the fact that the linear part in general would not reduce to the linear point interaction (e.g. in the case of [3] the linear part coincides with the free Laplacian). We expect that the local existence result and the conservation laws still hold true, but blow-up phenomena could possibly arise, if the nonlinear dependence of the coefficients is sufficiently strong.

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## References

[1] Adami R, Golse F and Teta A 2007 Rigorous derivation of the cubic NLS in dimension one *J. Stat. Phys.* **127** 1193–220

[2] Adami R and Sacchetti A 2005 The transition from diffusion to blow-up for a NLS equation in dimension one *J. Phys. A: Math. Gen.* **38** 8379–92

[3] Adami R and Teta A 2001 A class of nonlinear Schrödinger equations with concentrated nonlinearity *J. Funct. Anal.* **180** 148–75

[4] Akhiezer N and Glazman I 1963 *Theory of Linear Operators in Hilbert Spaces* (New York: Ungar)

[5] Albeverio S, Brzeźniak Z and Dabrowski L 1995 Fundamental solutions of the heat and Schrödinger equations with point interaction *J. Funct. Anal.* **130** 220–54

[6] Albeverio S, Gesztesy F, Hoegh-Krohn R and Holden H 1988 *Solvable Models in Quantum Mechanics* (New York: Springer)

[7] Albeverio S and Kurasov P 2000 *Singular Perturbations of Differential Operators* (Cambridge: Cambridge University Press)

[8] Bergé L 1999 Wave collapse in physics: principles and applications to light and plasma waves *Phys. Rep.* **303** 259–370

[9] Brazhnyi V, Konotop V and Perez-García V 2006 Driving defect modes of Bose Einstein condensates in optical lattices *Phys. Rev. Lett.* **96** 060403

[10] Brazhnyi V, Konotop V and Perez-García V 2006 Defect modes of a Bose Einstein condensate in an optical lattice with a localized impurity *Phys. Rev.* A **74** 023614

[11] Cao Xiang D and Malomed A B 1995 Soliton defect collisions in the nonlinear Schrödinger equation *Phys. Lett.* A **206** 177–82

[12] Cazenave T 2003 *Semilinear Schrödinger Equations (Courant Lecture Notes in Mathematics* vol 10*)* (Providence, RI: American Mathematical Society)

[13] Coclite G M and Holden H 2007 The Schrödinger–Maxwell system with Dirac mass *Ann. Inst. Henri Poincaré, Ann. Non Lin.* **24** 773–93

Coclite G M and Holden H 2008 The Schrödinger–Maxwell system with Dirac mass *Ann. Inst. Henri Poincaré, Ann. Non Lin.* **25** 833–66 (erratum)

[14] D'Ancona P, Pierfelice V and Teta A 2006 Dispersive estimates for the Schrödinger equation with point interactions *Math. Methods Appl. Sci.* **29** 309–23

[15] Della Casa F and Sacchetti A 2006 Stationary states for nonlinear one dimensional Schrödinger equation with singular potential *Physica* **209** 60–8

[16] Elander N and Kurasov P 1993 On the $\delta'$ *Interaction in One Dimension* MSI 93-7, ISSN-1100-214X (Stockholm, Sweden)

[17] Erdős L, Schlein B and Yau H-T 2007 Derivation of the cubic nonlinear Schrödinger equation from quantum dynamics of many body systems *Invent. Math.* **167** 515–614

[18] Erdős L, Schlein B and Yau H-T 2006 Derivation of the Gross–Pitaevskii equation for the dynamics of the Bose Einstein condensate *Ann. Math.* at press (xxx.lanl.gov/abs/math-ph/0606017)

[19] Exner P and Grosse P 1999 Some properties of the one-dimensional generalized point interactions (a torso), mp-arc 99-390, math-ph/9910029

[20] Fukuizumi R, Ohta M and Ozawa T 2008 Nonlinear Schrödinger equation with a point defect *Ann. Inst. Henri. Poincaré An. Non Lin.* **25** 837–45

[21] Goldstein J A 1985 *Semigroups of Linear Operators and Applications* (Oxford: Oxford University Press)

[22] Goodman R H, Holmes P J and Weinstein M I 2004 Strong NLS soliton-defect interactions *Physica* D **192** 215–48

[23] Holmer J 2007 Marzuola J, Zworski M Fast soliton scattering by delta impurities *Commun. Math. Phys.* **274** 187–216

[24] Kuchment P 2003 Quantum graphs: I. Some basic structures *Waves Random Complex Media* **14 1** S107–28

[25] Lieb E H, Seiringer R, Solovej J-P and Yngvason J 2005 *The Mathematics of the Bose Gas and Its Condensation* (Birkhauser: Basel, Berlin, Boston)

[26] Sandoval Romero M and Weder R 2006 The initial value problem, scattering and inverse scattering, for Schrödinger equations with a potential and a non-local nonlinearity *J. Phys. A: Math. Gen.* **39, 37** 11461–78

[27] Segal I 1962 Nonlinear semigroups *Ann. Math.* **78** 334–62

[28] Taylor M E 1996 *Partial differential equations, I, Basic Theory* (New York: Springer)

[29] Witthaut D, Mossmann S and Korsch H J 2005 Bound and resonance states of the nonlinear Schrödinger equation in simple model systems *J. Phys. A: Math. Gen.* **38** 1777–02